THE CRUSH-DOWN EQUATION
FOR NON-CONSTANT VELOCITY PROFILES

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ABSTRACT. Bažant et al. have proposed a model for a gravity-driven collapse of a tall building that collapses after column failure in a single storey. Therein the collapsing building is described by three distinct sections. The top section which consists of the part above the first failing storey, the middle section which is pushed from above by the top section and consists of compacted building material, and the part of the building below which is still undamaged. The middle part is gaining height during the collapse, the lower section is loosing height. The resulting equation of motion is called Crush-Down Equation.

In a first approach Bažant and Verdure used a constant velocity profile for the middle section, namely the top section and the middle section are assumed to have the same velocity. In a second approach by Bažant, Le, Greening and Benson this assumption is dropped and the model is slightly modified. However, their modifications are based on unphysical assumptions and lead to an erroneous version of the Crush-Down Equation.

We give a detailed account of how to implement a non-trivial velocity profile for the middle section and thereby derive a more accurate version of the Crush-Down Equation.

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1. Introduction

1.1. **Background.** On the 11th of September 2001 both of the Twin Towers of the World Trade Center in New York City were struck by an aircraft. The North Tower collapsed approximately one and a half hour after the aircraft impact, the South Tower approximately one hour after the aircraft impact. Both buildings collapsed rapidly in a top to down manner.

The National Institute of Standards and Technology released a report in 2005 whose objective was to explain how the collapse of the towers initiated [NIST]. However, they did not even try to explain how the collapse progressed.

Two years later, in 2007, a model was proposed by Bažant and Verdure to describe the gravity-driven collapse of a tall building as a progressive floor collapse [BaVe07]. The heart of this model is the so-called Crush-Down Equation that describes the downward movement of the crushing front of the collapsing building. Another year later Bažant, Benson, Greening and Le modified this model slightly [BLGB08].

In this paper we shall not be concerned about what can be learned from the model when empirical data of the actual collapsing buildings is taken into account. This is done elsewhere [Schn17b]. This paper is solely focused on the modifications that are made on the Crush-Down Equation in [BLGB08].

1.2. **The Crush-Down Equation.** Let us introduce some notation and the Crush-Down Equation of [BaVe07].

Consider a tall 1-dimensional building whose roof has an elevation of H over concourse level (cp. Figure 1). The total height of the building including its underground storeys will be denoted by $H_{tot} \geq H$. We fix a coordinate system which is pointing downwards to the ground and whose origin has a fixed elevation above concourse level, namely the elevation of the initial undestroyed tower top H.

Assume one storey of the building fails due to some extraordinary circumstances at the position $z_0 > 0$ below the tower top, i. e. on the interval $[z_0, z_0 + h]$ the column strength is reduced, where h is the height of one storey. The value of z_0 is the height of the top section of the building that will now decent and destroy
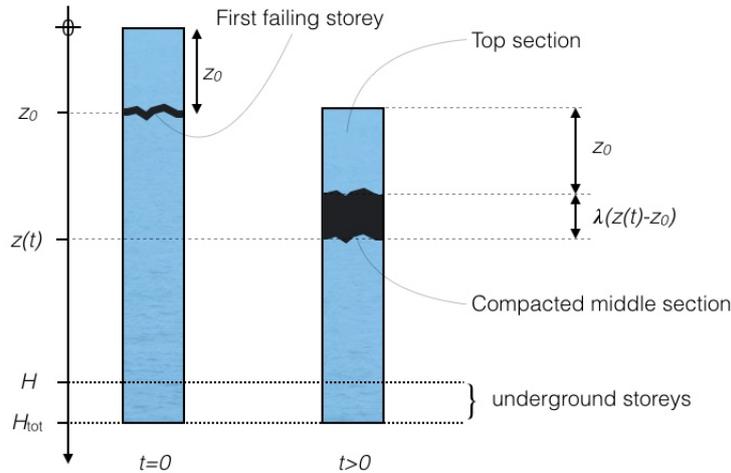

FIGURE 1. Schematic illustration of the gravity-driven collapse of a tall building.

the building below it. The destruction of the top section itself can be neglected. An argument for this assumption is given in [BLGB08, Appendix], where a two-sided front propagation is computed. The upward running front of destruction terminates within a fraction of a second after having propagated a couple of centimetres only.

Of course, we assume here that the dynamical load of the impacting top section is sufficiently big such that the collapse will indeed progress. A sufficient condition for the termination of the collapse is given in [BaZh02]. The estimate therin, however, is not optimal in the sense that a weaker condition already suffices to arrest the collapse [Schn17a].

The crushing front will run downwards and we denote the position of the crushing front at time t by $z(t) \geq z_0$. Time starts at collapse initiation, so $z(0) = z_0$.

We model the collapsing building by three distinct parts which are: 1. The initial top section of mass m_0 that sat above the first failing floor. This section keeps its height z_0 until the crushing front hits the ground. 2. The section below the top section which is compacted from its original undamaged size and moving with the same velocity as the top section. The height and mass of this section is growing in time. 3. The resting, still undamaged section below these two. The height of this section at time t is $H - z(t)$. It is reducing while the collapse proceeds.

We assume that during the collapse some fraction of material is spit outwards at the crushing front. We denote this quantity by $\kappa_{\text{out}} \in [0, 1]$. A value of $\kappa_{\text{out}} = 0$ is used in [BaVe07], $\kappa_{\text{out}} = 0.2$ in [BLGB08] and $\kappa_{\text{out}} = 0.25$ in [Schn17b].

Let us first assume all storeys are compacted to the same height after the crushing front has passed by. We describe this by the so-called compaction parameter $\lambda \in (0, 1)$, i.e. if $\Delta z(t) := z(t) - z_0$ is the height of the part of the building that has been crushed already, then $\lambda \cdot \Delta z(t)$ is the height of the compacted section at time t.

The position of the roof top at time t is $z(t) - \lambda \Delta z(t) - z_0 = (1 - \lambda)\Delta z(t)$, and its time derivative $(1 - \lambda)\dot{z}(t)$ is the downward velocity of both the top and the middle section. Therefore the total momentum of the falling two sections at time t is given by

$$(1) \qquad p(t) = m_0 \, (1 - \lambda) \, \dot{z}(t) + \Delta m\big(z(t)\big) \, (1 - \lambda) \, \dot{z}(t),$$

where

$$(2) \qquad \Delta m(z) := (1 - \kappa_{\text{out}}) \int_{z_0}^{z} \mu(x) \, dx$$

is the mass of the compacted section. $\mu(\cdot)$ is the mass height-density of the undestroyed tower. We will assume that μ is a strictly positive, monotonously increasing, convex continuous function. In particular, it follows that the mass function $\Delta m(\cdot)$ is also convex. Most of our theory can be done for general μ, but later we shall restrict ourselves to the case of the World Trade Center.

Now, the equation of motion—which is called Crush-Down Equation in [BaVe07]— that is valid until the crushing front reaches the ground is given by

$$(3) \qquad \frac{d}{dt} p(t) = m_0 \, g + \Delta m\big(z(t)\big) g - F\big(z(t)\big),$$

where $F(\cdot) > 0$ is the upward resistance force due to column buckling, and g is the acceleration of gravity.

Other (velocity dependent) forces can be added to F, but we shall not be concerned about this issue here. A discussion of the three terms that are added in [BLGB08] is given in [Schn17b, Section 1.4]. In this paper we shall discuss the question how the total momentum $p(t)$ changes, when two assumptions are altered. These are:

(*a*) The assumption that all storeys get compacted to the same height when the crushing front passes by.

(*b*) The assumption that the compacted section has a trivial (constant) velocity profile.

In [BLGB08] these two points are handled in the following way:

(*a*) Instead of assuming that every storey is compacted to the same height, it is assumed that every storey is compacted to the same density. This means that instead of $\lambda \in (0,1)$ a function $\lambda : [z_0, H_{\text{tot}}] \to (0,1)$ is considered such that λ is proportional to μ. This assumption simplifies the analysis significantly as we shall see, but it is not clear at all that this assumption is more realistic than the first one. It seems reasonable to expect that during the collapse the lower storeys get compacted to a higher density than the storeys above. A constant compaction parameter captures at least a glimpse of this aspect.

We shall consider both cases $\lambda = const$ and $\lambda \sim \mu$ in what follows. Note that these two cases coincide if $\mu(\cdot)$ is constant itself.

(*b*) The velocity profile of the middle section is supposed to vary linearly from the top of the middle section down to the crushing front. However, this modification is not done accurately in [BLGB08] for the following reasons:

  (*i*) If the velocity profile is non-trivial, then conservation of mass implies that the density of the compacted section is also varying. Yet in [BLGB08] it is assumed that the density is constant.

  (*ii*) The linear velocity profile of [BLGB08] is assumed to vary between the velocity of the top section (at the top of the compacted layer) and the velocity of the crushing front (at the bottom of the compacted layer). This is an extremely unphysical assumption, because the latter velocity is bigger than the first one. Realistically, the velocity at the bottom of the compacted layer is lower than the velocity at the top. The velocity of the crushing front should not be regarded as the velocity of any mass-bearing instance, but as a quantity that describes the change of the geometry of the crushing building.

The formula for the total momentum that is derived from these erroneous assumptions is formula (2) in [BLGB08]. It can be written as

$$(4) \qquad p(t) = m_0 \left(1 - \lambda(z)\right) \dot{z} + \Delta m(z) \left(1 - \frac{1}{2}\lambda(z)\right) \dot{z}.$$

Note that the second velocity term $(1 - \frac{1}{2}\lambda(z))\dot{z}$ is just the mean of $\dot{z}$ and $(1 - \lambda(z))\dot{z}$, i.e. that's the average velocity of the linear linear velocity profile.

2. CRUSH-DOWN FOR NON-TRIVIAL VELOCITY PROFILES

2.1. **The Set-Up.** Let us denote by $y(t)$ the position of the top of the middle section at time t. If $z(t)$ is the position of the crushing front at time t, the compacted layer has an extension of $\Lambda(t) := z(t) - y(t)$. In particular $\Lambda(0) = 0$. We shall derive a relation between Λ and the compaction parameter $\lambda : [z_0, H] \to (0,1)$. $\lambda(z(t))$ describes how much the storeys are compacted at the crushing front $z(t)$. For the two cases $\lambda = const$ and $\lambda \sim \mu$ the theory which we develop is manageable. To clarify our notation we shall use the convention $\lambda(z) = \lambda_0$ if λ is a constant function, $\lambda_0 \in \mathbb{R}$. If $\lambda \sim \mu$, we will use the convention $\lambda(z) = \lambda_0 \frac{\mu(z)}{\mu_0}$, where $\mu_0 := \mu(z_0)$.

By a we denote the distance from $y(t)$ to some point $y(t) + a$ in the compacted section, $a \in [0, \Lambda(t)]$. Let us now assume a non-trivial velocity profile $(a, t) \mapsto$

$v(a,t)$. $v(a,t)$ is the difference of $\dot{y}(t)$ and the velocity of the falling part at position $y(t) + a$ at time t. So the downward velocity of the falling section at position $y(t) + a$ and time t is given by $\dot{y}(t) + v(a,t)$. In particular $v(0,t) = 0$. If $v(a,t) = 0$ for all (a,t) we are in the situation as discussed before. The assumption of a linear velocity profile as done in [BLGB08] that reaches the velocity $\dot{z}(t)$ at at the crushing front means that the formula

$$(5) \qquad v(a,t) = \frac{a}{\Lambda(t)} \dot{\Lambda}(t)$$

holds. In fact, in that case $\dot{y}(t) + v(\Lambda(t),t) = \dot{y}(t) + \dot{\Lambda}(t) = \dot{z}(t)$. Let us now more generally assume that the velocity profile is of the form

$$(6) \qquad v(a,t) = \eta \cdot w\left(\frac{a}{\Lambda(t)}\right) \cdot \dot{\Lambda}(t),$$

for $\eta \in \mathbb{R}$ and $w : [0,1] \to [0,\infty)$ a positive, sufficiently smooth function such that $w(0) = 0, w(1) = 1$. Realistically, w is monotonously increasing and convex. The prototype of such a function is $w(x) = x^\nu$ for $\nu \geq 1$. $\nu = 1$ gives a linear velocity profile $a \mapsto v(a,t)$, but for all w we have $v(\Lambda(t),t) = \eta \dot{\Lambda}(t)$.

We shall not determine w explicitly. However, we shall derive some physical restrictions on η and w. In particular, we will see that $\eta \leq 0$ and $w'(0) = 0$. This excludes linear velocity profiles except the trivial one.

2.2. **Mass Conservation.** Let $m(a,t)$ be the mass of the falling section between the two points $y(t)$ and $y(t) + a$ at time t, $a \in [0, \Lambda(t)]$. We have the following boundary condition

$$(7) \qquad m(\Lambda(t),t) = \Delta m(z(t)), \quad \text{for all } t > 0.$$

Conservation of mass requires that the change of the mass $m(a,t)$ in time is given by the amount of mass that is moving into or out of $[y(t), y(t) + a]$, i.e.

$$(8) \qquad 0 = \partial_t m(a,t) + \partial_a m(a,t) \cdot v(a,t),$$

where $\partial_a m(a,t) =: \rho(a,t) > 0$ is the height-density at the point $y(t) + a$ at time t.

2.3. **The Effective Compaction Parameter.** Before we state the solution of (8) let us derive the relation between Λ and λ. We have by mass conservation

$$(9) \qquad \begin{aligned} \frac{d}{dt} m(\Lambda(t),t) &= \partial_a m(\Lambda(t),t)\dot{\Lambda}(t) + \partial_t m(a,t) \\ &= (\dot{\Lambda}(t) - \eta\dot{\Lambda}(t))\rho(\Lambda(t),t). \end{aligned}$$

On the other hand (7) implies that

$$(10) \qquad \frac{d}{dt} m(\Lambda(t),t) = (1 - \kappa_{\text{out}})\dot{z}(t)\mu(z(t)).$$

In the case of the linear velocity profile of [BLGB08] as stated in (5) we have $\eta = 1$. This implies $\kappa_{\text{out}} = 1$ which is physically absurd. So let us now continue in the case $\eta \neq 1$.

If $\lambda(z(t)) > 0$ is the (not necessarily constant) compaction ratio at the crushing front $z(t)$, then the density ρ satisfies $\rho(\Lambda(t),t) = (1 - \kappa_{\text{out}})\mu(z(t))/\lambda(z(t))$. So the two above equations imply

$$(11) \qquad \Lambda(t) = \frac{1}{1-\eta} \int_{z_0}^{z(t)} \lambda(x)\,dx.$$

Firstly, $\Lambda(t) > 0$, so $\eta < 1$. Secondly, if during the collapse some part of the building has been compacted, then it will not extend afterwards. Therefore $\Lambda(t)$ must be smaller than or equal to $\int_{z_0}^{z(t)} \lambda(x)\,dx$ which means the physically meaningful

range of η is $\eta \leq 0$ as we have claimed. We will restrict ourselves from now on to $\eta \leq 0$.

At this stage it should be noted that the velocity at the top of the compacted section is

$$(12) \qquad \dot{y} \;\; = \;\; \dot{z} - \dot{\Lambda} = \dot{z} - \frac{\lambda(z)}{1-\eta}\dot{z} = \Big(1 - \underbrace{\frac{\lambda(z)}{1-\eta}}_{=:\,\lambda^\dagger(z)}\Big)\dot{z}.$$

Because of the factor $1/(1-\eta)$ in (11) the above defined $\lambda^\dagger$ has—during the period of the Crush-Down—the rôle of λ in case of the trivial velocity profile ($\eta = 0$). We call $\lambda^\dagger$ the *effective compaction parameter*.

The velocity at the bottom of the crushed section is

$$(13) \qquad \begin{aligned} \dot{y}(t) + v(\Lambda(t),t) \;\; &= \;\; \big(\dot{z}(t) - \dot{\Lambda}(t)\big) + \eta\dot{\Lambda}(t) \\ &= \;\; \Big(1 - (1-\eta)\lambda^\dagger(z(t))\Big)\dot{z}(t). \end{aligned}$$

As this velocity must be positive, we find a further restriction to η. E. g. if $\lambda^\dagger(z) \geq 0.18$ for all z, we have $\eta \geq 1 - 1/0.18 \simeq -4.6$.

2.4. **The Derived Mass Distribution.** We will now specify $\lambda(\cdot)$ to be either $\lambda = const$ or $\lambda \sim \mu$, for in these cases we can easily solve the partial differential equation (8) with the boundary condition (7). This can be done by integrating the characteristic curves of (8). On the domain $\Omega := \{(a,t) : a \in (0, \Lambda(t)], t > 0\}$ the solution is[1]

$$(14) \qquad m(a,t) = \begin{cases} \Delta m\left(z_0 + \Delta z(t) \cdot f\left(\frac{a}{\Lambda(t)}\right)\right), & \text{for } \lambda = const, \\ \Delta m(z(t)) \cdot f\left(\frac{a}{\Lambda(t)}\right), & \text{for } \lambda \sim \mu, \end{cases}$$

where

$$(15) \qquad f(x) = \exp\left(-\int_x^1 \frac{db}{b - \eta\,w(b)}\right), \quad \text{for } x \in (0,1].$$

Of course, a physically meaningful solution should have an extension to $a = 0$ such that $m(0,t) = 0$ for all $t \geq 0$. So to proceed to the completed domain $\overline{\Omega} = \{(a,t) : a \in [0, \Lambda(t)], t \geq 0\}$ we extend f by $f(0) := 0$ and $m(0,0) := 0$. We need the following statement which we formulate for $\eta \neq 0$. $\eta = 0$ is the case of the trivial velocity profile, where $f(x) = x$.

Lemma 2.1. *Let $\eta < 0$. If $w : [0,1] \to \mathbb{R}$ is a positive, continuously differentiable, monotonously increasing function, such that $w(0) = 0$ and $w(1) = 1$, then*

(1) *f is a continuous, concave function $[0,1] \to [0,1]$, twice continuously differentiable on $(0,1]$ such that $f(x) = 0, f(1) = 1$.*
(2) *$f'(1) = (1-\eta)^{-1}$.*
(3) *$f(x) > x$ for $x \in (0,1)$*
(4) *If $f'(0)$ exists, then f' is continuous on $[0,1]$.*
(5) *If $f'(0)$ exists, then $w'(0) = 0$.*
(6) *If $f'(0)$ exists, then $f'(0) > 1$.*
(7) *$f'(0)$ either exists or $\lim_{x \searrow 0} f'(x) = +\infty$.*

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[1] Once the solution is found one can just verify it by differentiation. We leave that computation to the reader.

(8) *If for some $\nu > 1$, $w(x) \le x^\nu$ for all x in a small neighbourhood of 0, then $f'(0)$ exists.*

Proof. (1) The integral in (15) goes to $+\infty$ for $x \to 0$, so the limit of $f(x)$ is 0, and so f is continuous. Obviously, $f(x) \in [0,1]$ and $f(1) = 1$. Now observe that f satisfies the differential equation

$$f'(x) = \frac{f(x)}{x - \eta \, w(x)}, \quad \text{for } x > 0.$$

Differentiation gives

$$f''(x) = \frac{f(x) \, \eta \, w'(x)}{(x - \eta \, w(x))^2} \le 0, \quad \text{for } x > 0,$$

so f is concave on $(0,1]$. As f is continuous, f is concave on the compact interval. Obviously, f'' exists for $x > 0$ and is continuous.

(2) $f'(1) = (1 - \eta)^{-1}$ follows directly from the differential equation for f.

(3) As $\eta < 0$ it follows from (2) and concaveness that $f(x) > x$ for all $x \in (0,1)$.

(4) The differential equation for f can be rewritten as

$$f'(x) = \frac{f(x)}{x} \cdot \frac{1}{1 - \eta \frac{w(x)}{x}} \quad \text{for } x > 0.$$

Therefore so if $f'(0) = \lim_{x \searrow 0} f(x)/x$ exists, then also

$$\lim_{x \searrow 0} f'(x) = f'(0) \cdot \frac{1}{1 - \eta \, w'(0)}$$

exists. So $f'(0)$ and $\lim_{x \searrow 0} f'(x)$ both exist. Then they must be equal, for otherwise one could extend f differentiably to $x < 0$ by two different linear functions.

(5) If $f'(0)$ exists, then f' is continuous, so $f'(0) = f'(0) \cdot 1/(1-\eta \, w'(0))$ which implies $w'(0) = 0$.

(6) f is concave with $f(0) = 0$ and $f(1) = 1$, so $f'(0) \ge 1$. Then by (2) it follows $f'(0) > 1$.

(7) As f is concave, f' is monotonously decreasing, so $f'(0)$ does not exist if and only if $\lim_{x \to 0} f'(x) = +\infty$, because f' is continuous if $f'(0)$ exists.

(8) By (7) it suffices to show that $f(x)/(x - \eta \, w(x))$ is bounded. To do so assume first that $w(x) \le x^\nu$ for all x, let $h(x) := 1/(1 - \eta \, w(x)/x)$. h is a strictly positive, continuous function on the interval $(0,1]$ and it is bounded from above by 1. Then

$$
\begin{aligned}
\frac{f(x)}{x - \eta \, w(x)} &= \exp\left(-\int_x^1 \frac{1}{x - \eta \, w(x)} \, dx\right) \frac{1}{x} h(x) \\
&\le \exp\left(-\int_x^1 \frac{1}{x - \eta \, w(x)} \, dx\right) \frac{1}{x} \cdot 1 \\
&\le \exp\left(-\int_x^1 \frac{1}{x - \eta \, x^\nu} \, dx\right) \frac{1}{x} \\
&= \left(\frac{1 - \eta}{\left(\frac{1}{x}\right)^{\nu-1} - \eta}\right)^{\frac{1}{\nu-1}} \frac{1}{x} \\
&= \left(\frac{1 - \eta}{1 - \eta \, x^{\nu-1}}\right)^{\frac{1}{\nu-1}},
\end{aligned}
$$

and this is bounded. We leave it to the reader to modify the above estimate if $w(x) \le x^\nu$ only on a neighbourhood of 0. ∎

Realistically, the density $\rho(a,t) = \partial_a m(a,t)$ is a bounded positive function. Therefore we will assume that $f'(0)$ exists which by the above Lemma excludes linear velocity profiles from our discussion. Observe that the following formulas hold

$$(16) \quad \rho(a,t) = \begin{cases} \frac{(1-\kappa_{\text{out}}) \, \mu\left(z_0 + \Delta z(t) \cdot f\left(\frac{a}{\Lambda(t)}\right)\right)}{\lambda_0^\dagger} \cdot f'\left(\frac{a}{\Lambda(t)}\right), & \text{for } \lambda = const, \\ \frac{(1-\kappa_{\text{out}}) \, \mu_0}{\lambda_0^\dagger} \cdot f'\left(\frac{a}{\Lambda(t)}\right) & \text{for } \lambda \sim \mu. \end{cases}$$

Here we have set $\lambda_0^\dagger := \lambda^\dagger(0)$. Therefore in both cases at $a = 0$:

$$(17) \qquad\qquad \rho(0,t) = \frac{(1-\kappa_{\text{out}})\mu_0}{\lambda_0^\dagger} f'(0),$$

which is constant in time. We shall use this property of ρ to put a meaningful boundary condition on ρ in Appendix A.

For $t > 0$ let us denote by r_t ratio of the two densities $\rho(0,t)$ at the top of the compacted section and $\rho(\Lambda(t),t)$ at the crushing front. I. e. for all $t > 0$:

$$(18) \qquad r_t = \frac{\rho(0,t)}{\rho(\Lambda(t),t)} = \begin{cases} \frac{\mu(z_0)}{\mu(z(t))} \frac{f'(0)}{f'(1)}, & \text{for } \lambda = const, \\ \frac{f'(0)}{f'(1)}, & \text{for } \lambda \sim \mu, \end{cases}$$

We assume that the initial density μ is continuous and increasing towards the ground. So

$$(19) \qquad\qquad f'(0) = \sigma f'(1),$$

where $\sigma = \inf_t r_t \geq 1$, and by the above lemma we find for $\eta < 0$ that

$$(20) \qquad\qquad \eta > 1 - \sigma.$$

E. g. if we assume that during the collapse the variation of the density inside the compacted layer is below 50 %, i. e. $\sigma = 1.5$, we find that

$$(21) \qquad\qquad \eta > -0.5.$$

50 % should be regarded as a pretty high value for density variations in the compacted layer. Note that in [BLGB08] the approximation of a constant density in the compacted section is made.

2.5. **Estimates.** We continue with some technical aspects that will be useful later. All results in this section are obtained by elementary techniques.

Let f be given by (15) such that $f'(0) = \sigma f'(1)$. The first observation is only based on concaveness.

Lemma 2.2. *We have*

$$(22) \qquad\qquad \frac{1}{2} \leq \int_0^1 f(x)\,dx \leq \frac{\sqrt{\sigma}}{1+\sqrt{\sigma}}.$$

Proof. If $\eta = 0$, we have $f(x) = x$ and the lemma is trivially fulfilled as $\sigma = 1$. Now, assume $\eta < 0$. As f is concave the integral can be estimated by the unique piece-wise linear function $x \mapsto h_{\sigma,\eta}(x)$ that has slope $f'(0) = \sigma \cdot (1-\eta)^{-1}$ at $x = 0$ and slope $f'(1) = (1-\eta)^{-1}$ at $x = 1$ (two linear segments). The two segments are glued together at $x = \eta(1-\sigma)^{-1}$. We find

$$(23) \quad \int_0^1 f(x)\,dx \leq \int_0^1 h_{\sigma,\eta}(x)\,dx = 1 - \frac{1}{2}(1-\eta)^{-1}\left(1+\eta^2\,(\sigma-1)^{-1}\right),$$

and this expression is maximal for $\eta = 1 - \sqrt{\sigma}$. Inserting this value gives the lemma. ∎

Later we shall be interested in the term $q_0(f) := 2 \int_0^1 f(x)\,dx - 1$. For the above discussed value of $\sigma = 1.5$ we find $q_0(f) \leq 0.102$. For $\sigma = 1.3$ we have $q_0(f) \leq 0.066$, and for $\sigma = 1.1$ we have $q_0(f) \leq 0.024$.

To formulate the next lemma note that if f is given by (15), then the inverse function f^{-1} exists and is convex and monotonously increasing. Denote by $\widetilde{f} : [0,1] \to \mathbb{R}$ the positive function given by

$$(24) \qquad \widetilde{f}(y) = -\eta\, w(f^{-1}(y)).$$

As w and f^{-1} are monotonously increasing, so is $\widetilde{f}$. As the composition of two monotonously increasing, convex functions is again convex, $\widetilde{f}$ is convex. (As mentioned in Section 2.1, we assume w to be convex.)

Clearly, $\widetilde{f}(0) = \widetilde{f}'(0) = 0$, and $\widetilde{f}(1) = -\eta \leq \sigma - 1$.

Lemma 2.3. *We have* $q_0(f) = \int_0^1 \widetilde{f}(y)\,dy$.

Proof. Recall that by (15) f satisfies the differential equation

$$f'(x)\,(x - \eta\, w(x)) = f(x).$$

Therefore substituting $dy = f'(x)\,dx$ gives

$$
\begin{aligned}
\int_0^1 \widetilde{f}(y)\,dy &= \int_0^1 \Big(f(x) - x\,f'(x) \Big)\,dx \\
&= \int_0^1 f(x)\,dx - \Big[x\,f(x) \Big]_0^1 + \int_0^1 f(x)\,dx \\
&= 2\int_0^1 f(x)\,dx - 1
\end{aligned}
$$

which proves the lemma. ∎

Similar to Lemma 2.3 we find

$$(25) \qquad q_1(f) = \int_0^1 y\widetilde{f}(y)\,dy,$$

for $q_1(f) := \frac{1}{2}\left(3 \int_0^1 f(x)^2 dx - 1 \right)$.

The next statement is again a statement about convexity.

Lemma 2.4. *Define* $\gamma(f) \in \mathbb{R}$ *by* $q_0(f) = \gamma \cdot q_1(f)$, *then* $\gamma(f) \in [1, \frac{3}{2}]$.

Proof. Let g be a positive, convex, continuous function on $[0,1]$ such that $g(0) = 0, g(1) = 1$. Then $g(y) \leq y$, so $G(x) := \int_0^x g(y)\,dy \leq \frac{1}{2}x^2 \leq \frac{1}{2}$. G is also convex and satisfies

$$G(x) \leq G_0(x) := \begin{cases} \frac{1}{2}x^2, & \text{for } x \leq x_0 \\ x \cdot x_0 - \frac{1}{2}x_0^2, & \text{for } x > x_0, \end{cases}$$

where $x_0 := 1 - \sqrt{1 - 2\,G(1)}$. The linear part of G_0 is the tangent to the parabola that goes through $G(1)$ at $x = 1$.

Now, $G(1) = \int_0^1 g(y)\,dy$ and $G(1) - \int_0^1 G(y)\,dy = \int_0^1 y\,g(y)\,dy$. Therefore

$$
\begin{aligned}
\frac{\int_0^1 g(y)\,dy}{\int_0^1 y\,g(y)\,dy} &= \frac{G(1)}{G(1) - \int_0^1 G(x)\,dx} \\
&\leq \frac{G(1)}{G(1) - \int_0^1 G_0(x)\,dx} \\
&= \frac{G(1)}{\frac{2}{3}(\sqrt{1 - 2G(1)} - 1) + G(1)(2 - \frac{1}{3}\sqrt{1 - 2G(1)})}.
\end{aligned}
$$

This fraction is a monotonously increasing function of $G(1)$. So it is maximal for $G(1) = \frac{1}{2}$ (i. e. $g(y) = y$), which gives a value of $3/2$. Then the lemma follows for $g = \widetilde{f}/(-\eta)$. $\blacksquare$

Let g be a positive, continuous, convex function on $[0,1]$ such that $g(0) = 0$. We will only deal with $g(y) = y\,\widetilde{f}(y)$ and $g(y) = \widetilde{f}(y)$. For $z_1 > z_0$ consider the function

$$(26) \qquad g_\# : z \mapsto \theta(z - z_1) \int_{\frac{\Delta z_1}{\Delta z}}^{1} g(y)\, dy,$$

where $\Delta x := x - z_0$ and θ is the Heaviside step function which vanishes for negative arguments and is constant 1 for non-negative arguments. The following result is easy:

Lemma 2.5. *The function $g_\#$ is positive, monotonously increasing, continuous and for $z \geq z_1$ it is concave. Moreover $g'_\#(z_1) = \frac{1}{\Delta z_1} g(1)$ (right-sided derivative), and for $z \to \infty$ we have $g_\#(z) \nearrow \int_0^1 g(y)\, dy$.*

We want to approximate the function $g_\#$ by a piece-wise linear function, the red function in Figure 2, which is the mean of the indicated upper and lower bounds. To make that explicit note that in both cases $g = \widetilde{f}$ and $g = \mathrm{id} \cdot \widetilde{f}$ we have $g(1) = -\eta \leq \sigma - 1$. Now, the total hight of the building H_{tot} should be regarded as close to ∞ in the following sense: Let $\varepsilon_0 := \int_0^{\frac{\Delta z_1}{\Delta H_{\text{tot}}}} g(y)\, dy$. In our main application in the next subsection we will have $z_0 = 46\,\text{m}, z_1 = 110\,\text{m}$ and $H_{\text{tot}} = 438\,\text{m}$, so $\frac{\Delta z_1}{\Delta H_{\text{tot}}} < 0.164$, and because g is convex we find

$$(27) \qquad \varepsilon_0 < 0.164^2 \cdot \int_0^1 g(y)\, dy < 0.03 \cdot \begin{cases} q_0(f), & \text{for } g = \widetilde{f}, \\ q_1(f), & \text{for } g = \mathrm{id} \cdot \widetilde{f}. \end{cases}$$

We estimate the function $g_\#$ from above by

$$(28) \qquad g_\#(z) \ \leq \ g_+(z) := \theta(z - z_1) \min\left\{ \frac{z - z_1}{\Delta z_1}(\sigma - 1), \int_0^1 g(y)\, dy \right\}$$
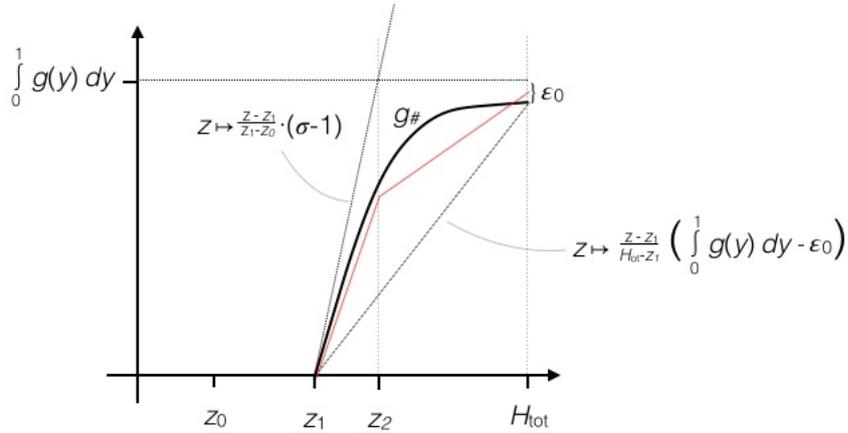

FIGURE 2. Approximating the concave part of the function $g_\#$.

and from below by

$$(29) \qquad g_{\#}(z) \;\; \geq \;\; g_-(z) := \theta(z - z_1) \frac{z - z_1}{H_{\text{tot}} - z_1} \left(\int_0^1 g(y)\, dy - \varepsilon_0 \right).$$

The maximal difference between g_+ and g_- is at the gluing point z_2 (Figure 2). We have $z_2 = z_1 + \frac{z_1 - z_0}{\sigma - 1} \int_0^1 g(y)\, dy$. The red function $g_{\text{red}} := \frac{1}{2}(g_+ + g_-)$ is the mean of g_+ and g_-. Therefore the maximal possible error between $g_\#$ and g_{red} is smaller than

$$(30) \qquad \frac{1}{2}(g_+(z_2) - g_-(z_2)) = \frac{1}{2} q \left(1 - \frac{q - \varepsilon_0}{\sigma - 1} \frac{z_1 - z_0}{H_{\text{tot}} - z_1} \right)$$

at the point z_2. Here $q = \int_0^1 g(y)\, dy$ which is either $q = q_0(f)$ or $q = q_1(f)$.

2.6. **The Modified Crush-Down Equation.** To derive the modified Crush-Down Equation $\frac{d}{dt} p(t) = m_0 g + \Delta m(z(t)) g - F(z(t))$ we just need compute the aggregated momentum $p(t)$ of the top section and the middle section. We have

$$
\begin{aligned}
(31) \quad p(t) \;\; &= \;\; m_0 \dot{y}(t) + \int_0^{\Lambda(t)} \rho(a,t)\,(\dot{y}(t) + v(a,t))\, da \\[2mm]
&= \;\; m_0 \dot{y}(t) + m(\Lambda(t), t)\,\dot{y}(t) - \int_0^{\Lambda(t)} \partial_t m(a,t)\, da \\[2mm]
&\overset{(12)}{=} \;\; (m_0 + \Delta m(z(t))) \left(1 - \lambda^\dagger(z(t)) \right) \dot{z}(t) - \int_0^{\Lambda(t)} \partial_t m(a,t)\, da.
\end{aligned}
$$

So the only structurally relevant change to the momentum given by (1) is the additional integral term. Note that $\lambda^\dagger$ appears instead of λ in (1).

Let us discuss the additional integral term in two cases $\lambda = \textit{const}$ and $\lambda \sim \mu$ separately. We start with the easier case of $\lambda \sim \mu$:

Proposition 2.6. *For $\lambda \sim \mu$ we have*

$$(32) \qquad \int_0^{\Lambda(t)} \partial_t m(a,t)\, da \;\; = \;\; q_0(f) \cdot \Delta m(z(t))\, \lambda^\dagger(z(t))\, \dot{z}(t),$$

where $q_0(f) = 2 \int_0^1 f(x)\, dx - 1$ as in (25).

Proof. The result follows from (14) after substituting $db = \frac{da}{\Lambda(t)}$ and by observing that

$$(1 - \kappa_{\text{out}})\,\mu(z)\,\dot{z}\,\Lambda = \Delta m(z)\,\lambda^\dagger(z)\,\dot{z}$$

and some integration by parts. ∎

In case of the trivial velocity profile ($\eta = 0$), we have $q_0(f) = 0$. For a non-trivial velocity profile we know a priori by Lemma 2.1 that $q_0(f) \in (0,1)$. However, by Lemma 2.2 we find a significantly lower upper bound for q once we impose physically reasonable assumptions on the density of the middle section. E. g. if we assume that the density variation in the compacted section is below 30 %, then $q_0(f) < 0.07$. To compare this result with the modified the Crush-Down Equation of [BLGB08] we write the total momentum as

$$(33) \qquad p(t) = m_0 \left(1 - \lambda^\dagger(z) \right) \dot{z} + \Delta m(z) \left(1 - (1 + q_0(f))\, \lambda^\dagger(z) \right) \dot{z}.$$

This formula has exactly the same structure as (4), where instead of $q_0(f) \in [0,1)$ a value of $q = -1/2$ is used. However, $q = -1/2$ is unphysical, because it has the wrong sign and its absolute value is far to big.

A nice property of the formula in Proposition 2.6 is that the time dependency (terms depending on z and $\dot{z}$) and the dependency of the velocity profile (the term $q_0(f)$) split into two separate factors. In the case of $\lambda = \textit{const}$ this is more involved,

because the function f appears in (14) in the argument of the mass function $\Delta m(\cdot)$. To state the result for the additional integral term in a manner similar to Proposition 2.6 let us introduce some short hand. For $x \in [z_0, z]$, define $\omega(x, z) \in \mathbb{R}$ by

$$(34) \qquad \Delta m'(x) = \frac{\Delta m(z)}{\Delta z} \cdot (1 + \omega(x, z)),$$

where $\Delta z = z - z_0$ as before. Apparently, ω measures how much the mass distribution Δm fails to be linear. Note that ω does not depend on $(1 - \kappa_{\text{out}})$, so ω indeed refers to the mass distribution of the undestroyed tower. Recall that μ is monotonously increasing, so $x \mapsto \omega(x, z)$ is also monotonously increasing. Note that by definition ω satisfies $\int_0^1 \omega(z_0 + \Delta z\, y, z)\, dy = 0$. Now, let

$$(35) \qquad Q(z, f) := \int_0^1 \omega(z_0 + \Delta z\, y, z)\, \widetilde{f}(y)\, dy \geq 0,$$

where $\widetilde{f}(y) = -\eta\, w(f^{-1}(y))$ as in (24). $Q(z, f)$ vanishes for a constant μ.

Proposition 2.7. *For $\lambda = const$ we have*

$$
\begin{aligned}
\int_0^{\Lambda(t)} \partial_t m(a, t)\, da &= \Lambda(t)\, \dot{z}(t) \int_0^1 \Delta m'(z_0 + \Delta z(t)\, y)\, \widetilde{f}(y)\, dy \\
&= \left(q_0(f) + Q(z(t), f) \right) \cdot \Delta m(z(t))\, \lambda_0^\dagger\, \dot{z}(t).
\end{aligned}
$$

Proof. The result follows from (14) after substituting first $dx = \frac{da}{\Lambda(t)}$, then $dy = f'(x)\, dx$, and then collecting the terms as in the proof of Lemma 2.3. ∎

$Q(z(t), f)$ has a time dependency, so we need to be a little careful about estimating its time derivative and the resulting term that appears in the Crush-Down Equation. To compute $\frac{d}{dt} p(t)$ for the Crush-Down Equation let us compute the time derivative of the additional integral term by Propostion 2.7. We have

$$(36)$$
$$
\begin{aligned}
\frac{d}{dt} \int_0^{\Lambda(t)} \partial_t m(a, t)\, da &= \lambda_0^\dagger\, \ddot{z}\, \Delta z \int_0^1 \Delta m'(z_0 + \Delta z\, y)\, \widetilde{f}(y)\, dy \\
&+ \lambda_0^\dagger\, \dot{z}^2 \Big(\int_0^1 \Delta m'(z_0 + \Delta z\, y)\, \widetilde{f}(y)\, dy \\
&\qquad + \Delta z \int_0^1 \Delta m''(z_0 + \Delta z\, y)\, y\, \widetilde{f}(y)\, dy \Big).
\end{aligned}
$$

Let us now deal explicitly with the mass distribution of the World Trade Center. In [Schn17b] the following formula is used

$$(37) \qquad \mu(x) = \mu_0 \cdot \left(1 + 0.43 \cdot \theta(x - z_1) \frac{x - z_1}{H - z_1} \right),$$

where θ is the Heaviside step function, $z_1 = 110\,\text{m}$, $H = 417\,\text{m}$ and $\mu_0 = 0.6 \cdot 10^6\,\text{kg/m}$. Using this explicit formula we can handle the two different integral terms of (36) using the notation of (26):

$$(38) \qquad \int_0^1 \Delta m'(z_0 + \Delta z\, y)\, \widetilde{f}(y)\, dy$$
$$= (1 - \kappa_{\text{out}})\mu_0 \left(q_0(f) + 0.43 \frac{\Delta z}{H - z_1} (\text{id} \cdot \widetilde{f})_\#(z) - 0.43 \frac{\Delta z_1}{H - z_1} \widetilde{f}_\#(z) \right),$$

and

$$(39) \qquad \Delta z \int_0^1 \Delta m''(z_0 + \Delta z\, y)\, y\, \widetilde{f}(y)\, dy = (1 - \kappa_{\text{out}})\mu_0 \cdot 0.43 \frac{\Delta z}{H - z_1} (\text{id} \cdot \widetilde{f})_\#(y).$$

Note that the right-hand side of (39) is the same as second summand in (38).

So far this result is exact, but as we do not have full information about the function f (or the actual velocity profile of the compacted section) we need to do some approximation. The numerical treatment of these terms can be tackled by approximating the functions $(\mathrm{id}\widetilde{f})_{\#}$ and $\widetilde{f}_{\#}$ by $(\mathrm{id}\widetilde{f})_{\mathrm{red}}$ and $\widetilde{f}_{\mathrm{red}}$ as explained in the previous section. To point out the essence of this approximation: A non-trivial velocity profile can be implemented in the Crush-Down Equation with three numerical parameters σ, $q_0(f)$ and $q_1(f)$ (or $\gamma(f)$ from Lemma 2.4). Note that in the beginning of the collapse, i.e. as long as $z(t) \le z_1$ this approximation is exact (in this range the two cases $\lambda = const$ and $\lambda \sim \mu$ coincide).

It is clear that one could give a precise estimate about error of the solution when $q_0(F), \sigma$ and $\gamma(f)$ vary. However, this is a tedious and lengthy and we will present for simplicity only a numerical treatment. This is more illustrative, and for practical reasons this is sufficient in any case.

To obtain a numerical solution $t \mapsto z(t)$ of the Crush-Down Equation we reformulate it as usual as a 2-dimensional differential equation of 1st order.

$$(40) \qquad \frac{d}{dt}\begin{pmatrix} z(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ \phi(z(t)) - \psi(z(t))\,v(t)^2 \end{pmatrix}.$$

The lengthy formulas for the coefficients $\phi(\cdot)$ and $\psi(\cdot)$ are explicitly stated in Appendix B, where the source code for a numerical solution of the Crush-Down Equation is given. We use the open-source computer algebra system Maxima (wx-Maxima 16.04.0) for solving (40) with an implementation of the Runge-Kutta algorithm. For the computation we use the same upward resistance force $F(z) = \chi(z)F_0(z)$ as in [Schn17b]. $\chi : [0, H_{\mathrm{tot}}] \to [0, 1]$ describes the damage of the building, and F_0 is the upward force of the undamaged columns. Here we use a energy dissipation of 250 MJ per storey, see [Schn17b, Sec. 1.2] for a discussion and details. The explicit formulas are also stated in Appendix B. For κ_{out} we use a value of 0.25 as in [Schn17b] and for $\lambda_0^{\dagger}$ we use a value of 0.13 which seems to be reasonable in view of the considerations in Appendix A.

Figure 3 shows the downward movement of the roof according to different solutions of the Crush-Down Equation. All the diagrams to the right show a zoom into the last second of the corresponding diagram to the left. We are interested only in the first ten seconds of the solution, because this is the time interval for which empirical data are given in [Schn17b].

In the top row the case of $\lambda \sim \mu$ for four different values of $q = q_0(f)$ including the unphysical value of $q = -1/2$ is computed. Note that in the realistic range the effect of q is extremely small. A value of $q = q_0(f) = 0.2$ requires by Lemma 2.2 that $\sigma \ge 2.25$, which means that the density of the compacted layer varies by over 125 %. Note also that the decent is faster if a non-trivial velocity profile is assumed, whereas the erroneous assumption of [BLGB08] lead to slower decent of the building.

The case of $\lambda = const$ is shown in the second row. The case $q = 0$ is the solution of the original unmodified Crush-Down equation.

The two diagrams in the third row compare the two cases $\lambda = const$ and $\lambda \sim \mu$ explicitly. It shows the magenta solution from the second row and the green solution from the top. Apparently the difference is extremely small compared to any observable quantity in a realistic collapse scenario.

The diagrams at the bottom compare the solution from the second row for $q = 0.1$ (magenta) with the unmodified solution $q = 0$ but with a little lower energy dissipation per storey due to column buckling (245 MJ instead of 250 MJ). We find
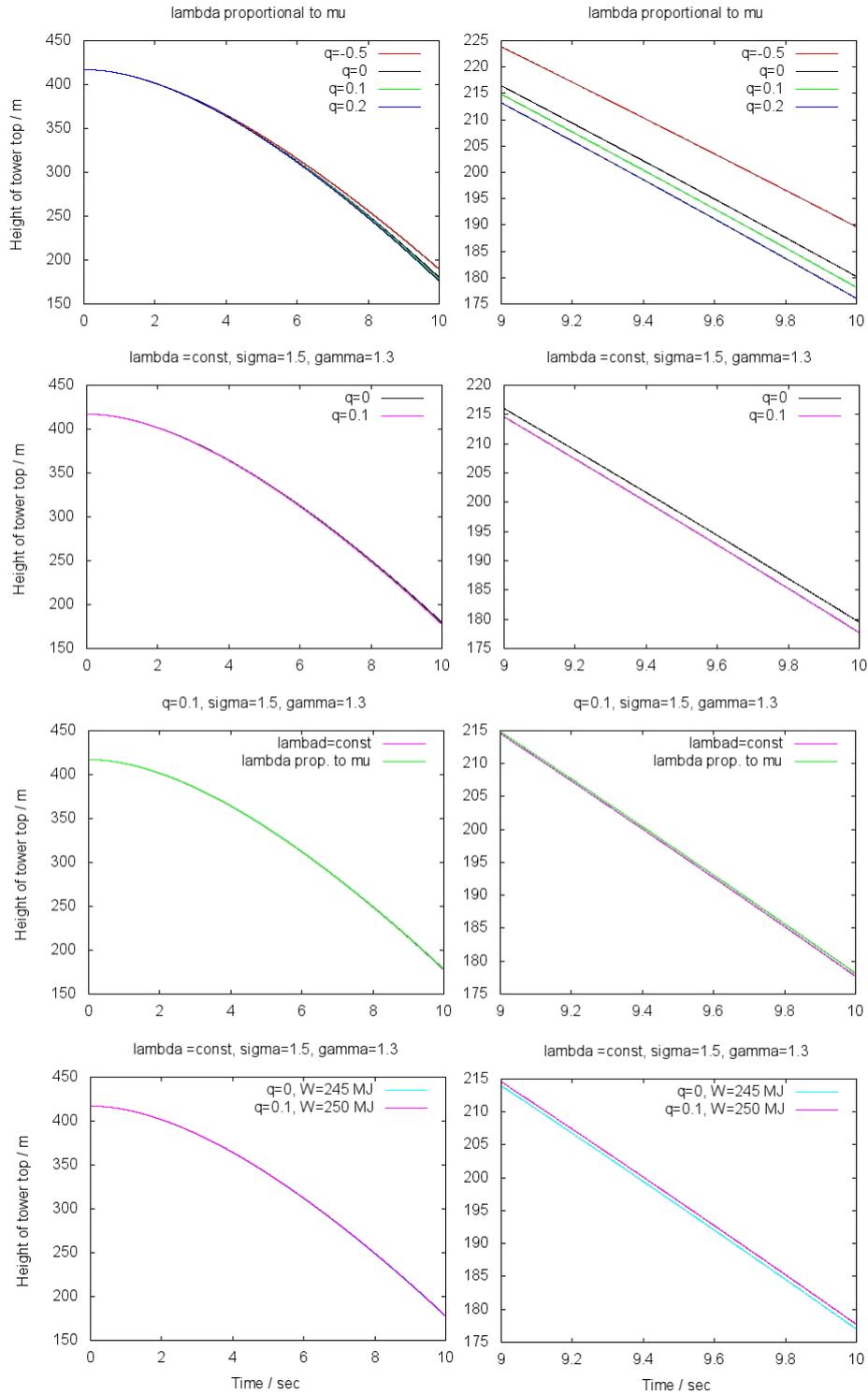
FIGURE 3. Comparing the solutions of the Crush-Down Equation.

that the effect of a non-trivial velocity profile is smaller than a tiny variation of 2% of the energy dissipation per storey.

3. Discussion of Results

3.1. **The Modified Crush-Down Equation.**
In [BLGB08] a modified version of the original Crush-Down Equation [BaVe07] is used which contains some unphysical assumptions about the compacted middle section of the crushing building.

We have corrected these modification in a more realistic set-up. The erroneous assumptions of [BLGB08] lead to a wrong prediction of the movement of the descending building. For the erroneous assumptions the predicted decent is slower whereas under realistic assumptions the decent is slightly faster.

3.2. **Conclusion.**
In [Schn17b] the collapse of the North Tower of the World Trade Center is analysed using a constant velocity profile for the compacted middle section. This assumption is fully justified by the presented results, because the uncertainty of the original parameters and the uncertainty of the of the measurements in [Schn17b] is too big to detect the effects of a non-trivial velocity profile.

Note that the main result of [Schn17b] is the enormous fluctuation of energy dissipation during the collapse. Because the building's predicted descent is faster for a non-trivial velocity profile, this result would be even bigger if a non-trivial velocity profile would be taken into account.

Appendix A. The Average Density of Rubble and the Total Compaction Parameter

Once the crushing front reaches the ground a further compaction of the compacted section will take place during the subsequent Crush-Up phase. The crushing front will start moving upwards from the ground through the middle section. Below the upward propagating front the movement has come to rest. Once the front reached the top of the compacted section it will start moving through the top section.

This consideration shall give us a boundary condition for density ρ at $a = 0$. To be consistent with our two cases $\lambda = const$ and $\lambda \sim \mu$ we derive two different boundary conditions.

If $\lambda \sim \mu$, then during the Crush-Down every storey is compacted to the same density $\rho(\Lambda(t), t) = (1 - \kappa_{\text{out}})\mu_0/\lambda_0$ at the crushing front. At the top of the compacted section the density is $\rho(0, t) = (1 - \kappa_{\text{out}})\mu_0\sigma\lambda_0$ constant in time and we shall require that below the Crush-Up front all of the building is compacted to $\rho(0, t)$. This density should therefore coincide with μ_c, the average density of the rubble pile. So our boundary condition is

$$(41) \qquad \mu_c \overset{!}{=} \rho(0, t) = (1 - \kappa_{\text{out}})\frac{\mu_0 \sigma}{\lambda_0}.$$

In [BLGB08, p. 895] a value of $\mu_c = 4.1 \cdot 10^6 \, \text{kg/m}$ is stated without reference as "*typical density of rubble*" not specifying whether this means *the* rubble of the Twin Towers or some observed rubble density of other building collapses. Using this value we find for $\mu_0 = 0.6 \cdot 10^6 \, \text{kg/m}$, $\sigma = 1.5$, and $\kappa_{\text{out}} = 0.25$:

$$(42) \qquad \lambda_0 = (1 - \kappa_{\text{out}})\frac{\mu_0}{\mu_c}\sigma = 0.16.$$

By (20) this implies

$$(43) \qquad \lambda_0^\dagger > (1 - \kappa_{\text{out}})\frac{\mu_0}{\mu_c} = 0.11.$$

In the case of $\lambda = const$ we assume that at the end of the Crush-Up the crushed building as a density distribution of[2]

---

[2] For simplicity we ignore a discussion about what happens to the parameter κ_{out} when the crushing front is moving through the underground storeys.

$$(44) \qquad \rho^{\ddagger}(b) = \frac{(1 - \kappa_{\text{out}})\mu\left(\frac{b}{\lambda^{\ddagger}}\right)}{\lambda^{\ddagger}},$$

where $b \in [0, \lambda^{\ddagger} \cdot H_{\text{tot}}]$ runs from the top of the collapsed building down to the ground. $H_{\text{tot}} = H + 21\,\text{m} = 438\,\text{m}$ includes 21 m of underground storeys. $\lambda^{\ddagger} \in \mathbb{R}$ is the *total compaction parameter* defined by $\frac{(1 - \kappa_{\text{out}})\,m_{\text{tot}}}{H_{\text{tot}}} = \lambda^{\ddagger} \cdot \mu_c$, where $m_{\text{tot}} = 288 \cdot 10^6\,\text{kg}$ is the total mass of the tower.[3] For $\kappa_{\text{out}} = 0.25$ this leads to a total compaction parameter of

$$(45) \qquad \lambda^{\ddagger} = 0.11.$$

The boundary condition we impose on ρ is

$$(46) \qquad \rho^{\ddagger}(z_0 \lambda^{\ddagger}) \stackrel{!}{=} \rho(0, t) = (1 - \kappa_{\text{out}})\frac{\mu_0\,\sigma}{\lambda_0},$$

which for the above mentioned numerical value of $\sigma = 1.5$ gives

$$(47) \qquad \lambda_0 = \sigma \cdot \lambda^{\ddagger} = 0.18$$

and again by (20)

$$(48) \qquad \lambda_0^{\dagger} > \lambda^{\ddagger} = 0.11.$$

The inequalities become equalities for the trivial velocity profile, i.e. $\eta = 0$.

APPENDIX B. COMPUTING NUMERICAL SOLUTIONS WITH MAXIMA

The following is the Maxima source code which we have used to compute the solutions of Figure 3. Note for the computation that the mass density and the upward column force miss a factor 10^6. However, this factor cancels out in the coefficients ϕ and ψ, so the solution is not effected by this simplification.

```
/* [wxMaxima:  input start] */
/* [Define the constants (lambda_0 := λ₀†, v_0 = initial veolocity)] */

g:9.8; H:417; h:3.8; z_0:46; z_1:110; v_0:0; mu_0:0.6; lambda_0:0.13; kappa:0.25;

/* [The Heaviside step function] */

theta(z):=if z<0 then 0 else 1;

/* [The damage function and the upward resistance force] */

chi(z):=(0.5+0.4*theta(z-z_0-h)+0.1*theta(z-z_0-4*h));
F(z):= 250/h *chi(z)*(1+theta(z-z_1)*(6*(z-z_1)/(H-z_1)));

/* [The mass density, and the masses m_0 := m₀ and Dm(z) := Δm(z) ] */

mu(z):= mu_0*(1+theta(z-z_1)*(0.43*(z-z_1)/(H-z_1))) ;
m_0:mu_0*z_0;
Dm(z):= (1-kappa)*mu_0*(z-z_0+ theta(z-z_1)*0.215*(z-z_1)^2/(H-z_1));

/* [For λ ∼ μ we need %lambda(z) := λ†(z) and its derivative dlambda(z).]  */
```

---

[3] $m_{\text{tot}} = 288,000\,\text{t}$ is the value that has been estimated meticulously in [Uric07]. In [BLGB08] a value of 500,000 t is stated without reference which would give $\lambda^{\ddagger} = 0.21$.

```
/* [Lambda(z) = the height of the compacted section = Λ(t)] */


%lambda(z):=lambda_0*mu(z)/mu_0;

dlambda(z):=lambda_0*theta(z-z_1)*0.43/(H-z_1);

Lambda(z):=lambda_0*Dm(z)/(mu_0*(1-kappa));


/* [The coefficients of the Crush-Down Equation for λ ~ μ] */


phi(z,q):=( (m_0+Dm(z))*g-F(z)) / ( m_0*(1-%lambda(z)) + Dm(z)*(1-(1+q)*%lambda(z)) );

psi(z,q):=( (1-kappa)*mu(z)*(1-(1+q)*%lambda(z)) - (m_0+Dm(z)*(1+q))*dlambda(z) )
          / ( m_0*(1-%lambda(z)) + Dm(z)*(1-(1+q)*%lambda(z)) );


/* [Four choices of q := q_0(f)] */


q_1:-0.5;

q_2:0;

q_3:0.1;

q_4:0.2;


/* [Compute the solutions for λ ~ μ] */


time:10;

stepwidth:0.001;

solution_1:rk([u, phi(z,q_1)-u^2*psi(z,q_1)],[z, u],[z_0,v_0],[t,0,time,stepwidth])$

solution_2:rk([u, phi(z,q_2)-u^2*psi(z,q_2)],[z, u],[z_0,v_0],[t,0,time,stepwidth])$

solution_3:rk([u, phi(z,q_3)-u^2*psi(z,q_3)],[z, u],[z_0,v_0],[t,0,time,stepwidth])$

solution_4:rk([u, phi(z,q_4)-u^2*psi(z,q_4)],[z, u],[z_0,v_0],[t,0,time,stepwidth])$


/* [Turn the solutions into the position of the roof (here we need the quantity Lambda(z))] */


height_1:makelist([solution_1[i][1],H-(solution_1[i][2]-Lambda(solution_1[i][2])-z_0)],i,1,length(solution_1))$

height_2:makelist([solution_2[i][1],H-(solution_2[i][2]-Lambda(solution_2[i][2])-z_0)],i,1,length(solution_2))$

height_3:makelist([solution_3[i][1],H-(solution_3[i][2]-Lambda(solution_3[i][2])-z_0)],i,1,length(solution_3))$

height_4:makelist([solution_4[i][1],H-(solution_4[i][2]-Lambda(solution_4[i][2])-z_0)],i,1,length(solution_4))$


/* [Plot the solutions for λ ~ μ] */


wxplot2d( [[discrete,height_1],[discrete,height_2],[discrete,height_3],[discrete,height_4]],
           [x,0,time],
           [style,[lines,1,red],[lines,1,black],[lines,1,green],[lines,1,blue]],
           [ylabel,"Height of tower top / m "],
           [xlabel,"Time / sec"],
           [title,concat("lambda proportional to mu")],
           [legend,concat("q=",string(q_1)), concat("q=",string(q_2)),
                   concat("q=",string(q_3)), concat("q=",string(q_4))]
)$


/* [For λ = const we need the approximation of the function g_# by g_red] */


epsilon:0.03;

g_plus(z,sigma,q):=theta(z-z_1)*min((z-z_1)/(z_1-z_0)*(sigma-1),q);

g_minus(z,q):=theta(z-z_1)*((z-z_1)/(H+21-z_1))*(1-epsilon)*q ;
```

```
g_red(z,sigma,q):=1/2*(g_plus(z,sigma,q)+g_minus(z,q));



/* [The coefficients of the Crush-Down Equation for λ = const] */
/* [gamma := γ(f) is from Lemma 2.4] */


Phi(z,sigma,q,gamma):=( (m_0+Dm(z))*g-F(z)) / ( (m_0+ Dm(z))*(1-lambda_0) - lambda_0*(z-z_0)*(1-kappa)*mu_0
                  *(q + 0.43*(z-z_0)/(H-z_1)*g_red(z,sigma,q/gamma) - 0.43*(z_1-z_0)/(H-z_1)*g_red(z,sigma,q)) );
Psi(z,sigma,q,gamma):=( (1-kappa)*mu(z)*(1-lambda_0) - lambda_0*(1-kappa)*mu_0
                  *(q+ 2*0.43*(z-z_0)/(H-z_1)*g_red(z,sigma,q/gamma) -0.43*(z_1-z_0)/(H-z_1)*g_red(z,sigma,q)) )
        / ( (m_0+ Dm(z))*(1-lambda_0) - lambda_0*(z-z_0)*(1-kappa)*mu_0
                  *(q+ 0.43*(z-z_0)/(H-z_1)*g_red(z,sigma,q/gamma) -0.43*(z_1-z_0)/(H-z_1)*g_red(z,sigma,q)) );


/* [Compute the solutions for λ = const] */

sigma:1.5;
gamma:1.3;
time:10;
stepwidth:0.001;
solution_a:rk([u*theta(u), Phi(z,sigma,q_2,gamma)-u^2*Psi(z,sigma,q_2,gamma)],[z, u],[z_0,v_0],
[t,0,time,stepwidth])$
solution_b:rk([u*theta(u), Phi(z,sigma,q_3,gamma)-u^2*Psi(z,sigma,q_3,gamma)],[z, u],[z_0,v_0],
[t,0,time,stepwidth])$



/* [Turn the solutions into the position of the roof] */

height_a:makelist([solution_a[i][1],H-(1-lambda_0)*(solution_a[i][2]-z_0)],i,1,length(solution_a))$
height_b:makelist([solution_b[i][1],H-(1-lambda_0)*(solution_b[i][2]-z_0)],i,1,length(solution_b))$



/* [Plot the solutions for λ = const] */

wxplot2d( [ [discrete,height_a],[discrete,height_b]],
          [x,0,time],
          [style,[lines,1,black],[lines,1,magenta]],
          [ylabel,"Height of tower top / m "], [xlabel,"Time / sec"],
          [title,concat("lambda =const, sigma=",string(sigma),", gamma=",string(gamma))],
          [legend, concat("q=",string(q_2),""),concat("q=",string(q_3),"")]
)$


/* [Compare the solutions for λ ∼ μ and λ = const ] */

wxplot2d( [ [discrete,height_b],[discrete,height_3]],
          [x,9,time],[style, [lines,1,magenta],[lines,1,green]],
          [ylabel,"Height of tower top / m "], [xlabel,"Time / sec"],
          [title,concat("q=",string(q_3),", sigma=",string(sigma),", gamma=",string(gamma))],
          [legend,concat("lambad=const"),concat("lambda prop.  to mu")]
)$


/* [wxMaxima:  input end] */
```

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